

# Evaluation Of Fast Silhouette Method For Determination Of Optimal Number Of Clusters Within A Sensor Network Coverage Area

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**Abstract-** In this work, the evaluation of Fast Silhouette method for determination of optimal number of clusters within a sensor network coverage area is presented. This work considered three approaches which are; Elbow method, Gap statistics method, and Silhouette method. The major limitation of Silhouette method, just like Elbow method and Gap statistics method is the requirement of initial centroids and number of clusters which plays significant role in the computational time. Accordingly, in this work an Enhanced Silhouette method is presented which has a faster computation speed. The sensor network coverage area is modelled as a rectangular area with length  $L(x) = 800m$ , width  $L(y) = 800m$  hence, area of  $800m \times 800m = 640,000 m^2$ . The simulation was done with 500 sensor nodes in the network. The results show that the fast Silhouette method has the lowest execution time of 2.7 seconds. The results of the normalized model execution time with respect to the Fast Silhouette time (expressed in %) shows that the Elbow method takes 215 % of the time required by the Fast Silhouette, the Gap statistics method takes 196% of the time required by the Fast Silhouette while the classical Silhouette method takes 201% of the time required by the Fast Silhouette. In all, the Fast Silhouette method realized the optimal number of cluster with execution time that is about half of the time required by the other three methods studied in this work.

**Keywords—** Silhouette Method, Wireless Sensor Network, Clustered Network, Gap Statistics Method Elbow Method, Network Coverage Area

## 1.0 INTRODUCTION

Nowadays, wireless sensor network (WSN) has become widely applied in diverse areas [1,2]. In some cases, the number of sensors required are so many or the spatial distribution of the sensors is such that clustering of the sensors is required to optimize the overall energy consumption and network lifespan [3,4,5]. In such cases, the optimal number of clusters is needed before the clustering algorithm can be applied [6,7].

Notably, in order to obtain the optimal performance in a clustered sensor network, the gateways must be optimally located within the sensor network coverage area based on the spatial distribution of the sensor nodes [8,9]. First, the number of clusters to be deployed must be optimally determined. The determination of the optimal number of clusters can be handled by various approaches. This work considered three of these approaches which are: Elbow method, Gap statistics method, and Silhouette method [10,11,12]. In addition, an enhanced Silhouette method is presented which has a faster computation speed. The details of each of the methods are presented in this work and simulation program written in Python programming language is used to implement each

of the methods and also compare the performance of the various methods.

## 2.0 METHODOLOGY

The approach employed in this work is to first present the network model. Then use the network model for the determination of optimal number of clusters using each of the four methods considered in this work.

### 2.1 The Network model

The sensor network coverage area is modelled as a rectangular area with length ( $L_x$ ), width ( $L_y$ ) and area  $A_{xy}$  and the model procedure is presented in Algorithm 1, while the model layout is presented in Figure 1.

### Algorithm 1: Network modelling procedure

- 1: **Begin**
- 2: Define number of sensor nodes  $N_{nodes}$ ,
- 3: Define the length and width parameters  $L_x, L_y$  of the farm,
- 4: Define the application area  $A_{xy}$  based on  $A_{xy} = L_x * L_y$
- 5: Generate and distribute random coordinates  $(x_k, y_k)$  using  $random()$  function. Coordinates points must be equal to  $N_{nodes}$ ,
- 6: Output the scatter plot,
- 7: **End**

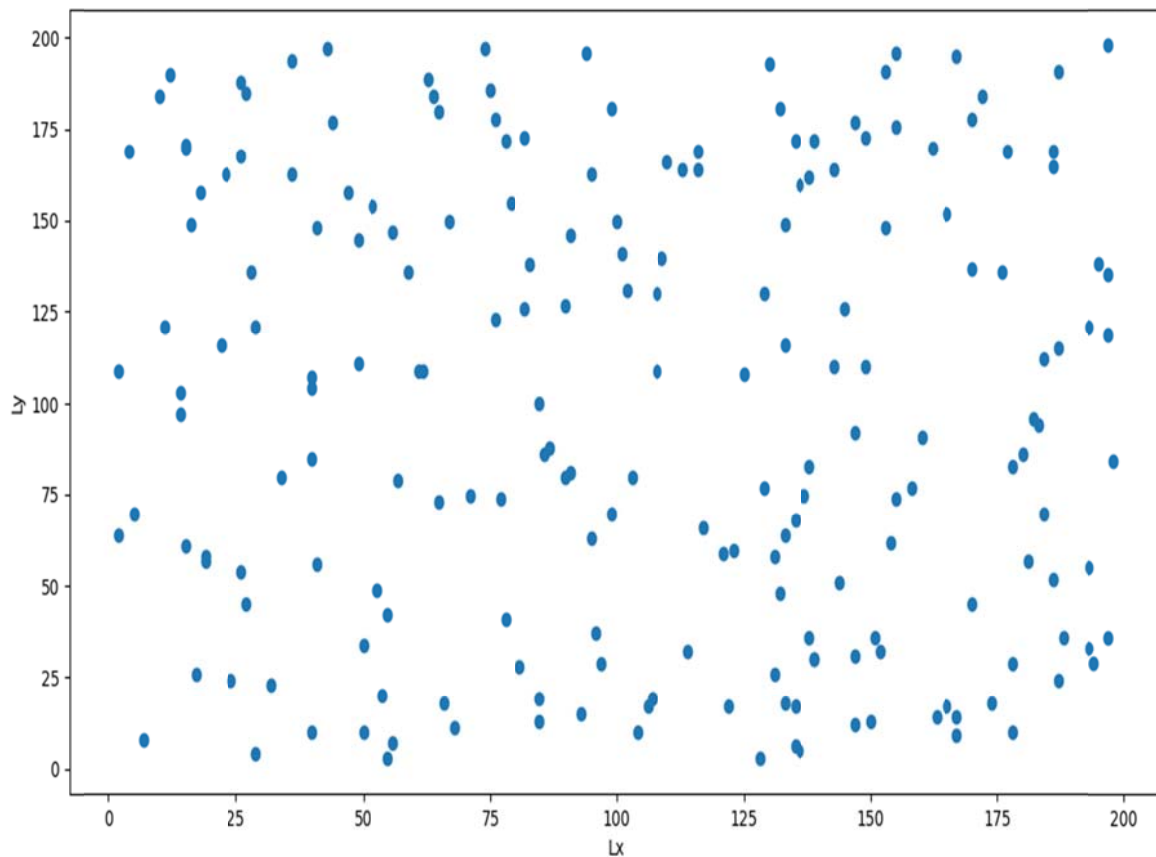


Figure 1: Network model

### 2.2 Determination of Optimal Number of Clusters using Elbow Method

The Elbow method for determination of optimal number of cluster is based on unsupervised machine learning scheme. In the Elbow method, the within-cluster sum of square (WCSS) is the major criteria used in determining the number of clusters based on Elbow method [13,14,15]. Let the location of the centroid of the cluster be denoted as  $(x_c, y_c)$ , the location of a sensor within the network space be denoted as  $(x_i, y_i)$ , and  $N_{clusters}$  denotes the total number of clusters, then  $WCSS$  can be computed by taking the Euclidean distance between the location of the centroid of the cluster and the location of the sensor node as shown in Equation (1);

$$WCSS = \sqrt{(x_c - x_i)^2 + (y_c - y_i)^2} \quad (1)$$

Typically,  $WCSS$  is inversely proportional to number of clusters as shown in Equation 2;

$$WCSS \propto \frac{1}{N_{clusters}} \quad (2)$$

Where  $N_{clusters}$  denotes number of clusters. This expression implies that  $WCSS$  shrinks as the number of clusters increases. Invariably,  $WCSS$  is maximum at  $k = 1$ , and minimum at  $k = N_{max}$ , where  $N_{max}$  denotes the maximum possible number of clusters. The procedure for determining the number of clusters based on Elbow method is presented in Algorithm 2.

### Algorithm 2: Determination of optimal number of clusters based on Elbow method

```

1: Begin
2: Choose  $k$  (the number of clusters)
3: Choose the number of centroid  $N_c = k$ 
4: Compute  $WCSS$  and obtain the new centroid
5: Use the new centroid to reallocate the entire coordinates
6: if the coordinates remain constant after reallocation of
   the coordinates on the dataset then
7:   goto 10
8: else

```

```

9:   goto 4
10:end

```

By plotting the  $WCSS$  with respect to number of clusters the optimal number of clusters can be obtained from Elbow method. For instance, as shown in Figure 2, it is observed that when the number of cluster is four, the line changes abruptly to have a near parallel comparison with the  $x$  - axis.

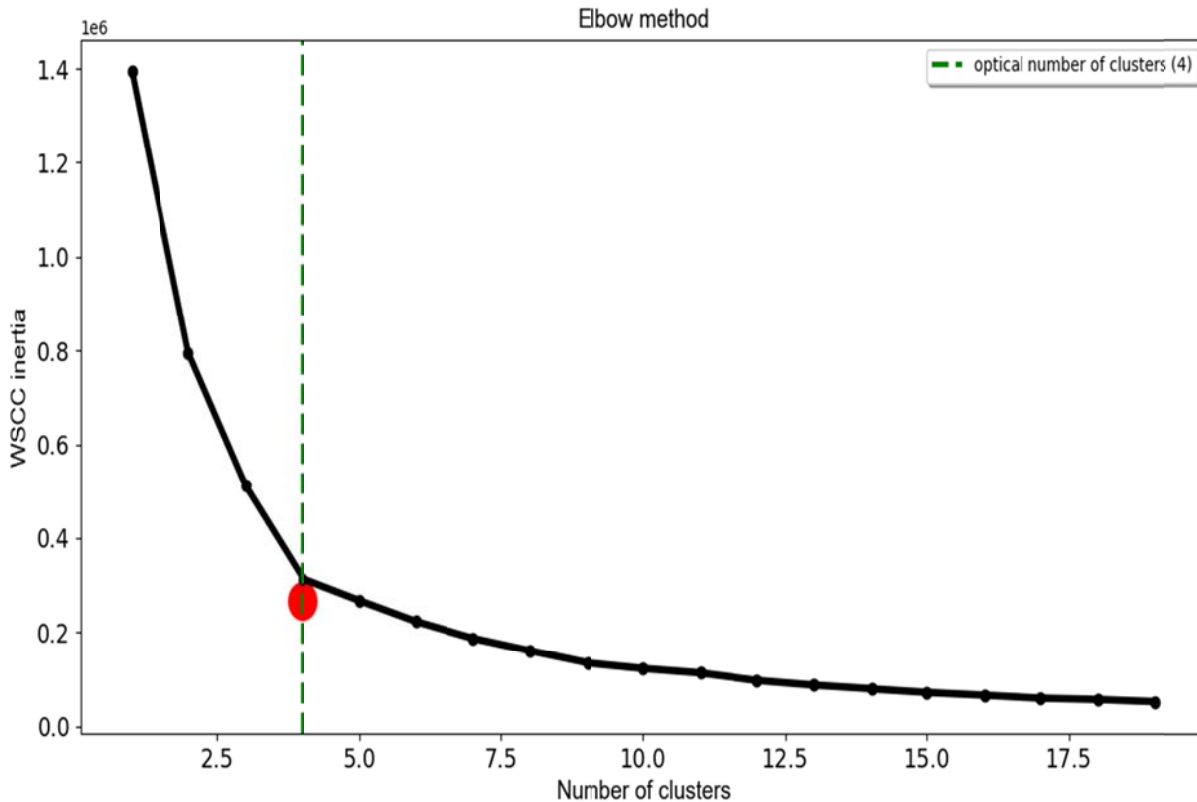


Figure 2: Optimal number of clusters using Elbow method

### 2.3 Determination of Optimal Number of Clusters using Gap Statistics Method

Gap statistics can be computed using Equation (3) [16,17];

$$Gap_n(k) = \sum_n \{\log(\delta_k)\} - \log(\delta_k) \quad (3)$$

Where,  $\delta_k$  denotes the degree of clustering based on  $WCSS$  which can be computed from Equation (4);

$$WCSS = \sum_{x_i \in C_k} \sum_{y_i \in C_k} \|x_i - y_i\|^2 \quad (4)$$

The procedure for determining the optimal number of cluster using gap statistics is presented in Algorithm 3 which shows that  $WCSS$  in this case is computed using the  $inertia\_property$  of K - Means method. For instance, applying Gap statistics method on a given system model

shows that the optimal number of clusters is 4 as shown in Figure 3.

### Algorithm 3: Determination of optimal number of clusters based on Gap Statistics method

```

1: Begin
2: Define null reference
3: Compute the cluster congestion
4: Group the reference data set with different number of
   clusters
5: Compute the congestion average on the dataset
6: Compute Gap statistics based on Equation (3).
7: End

```

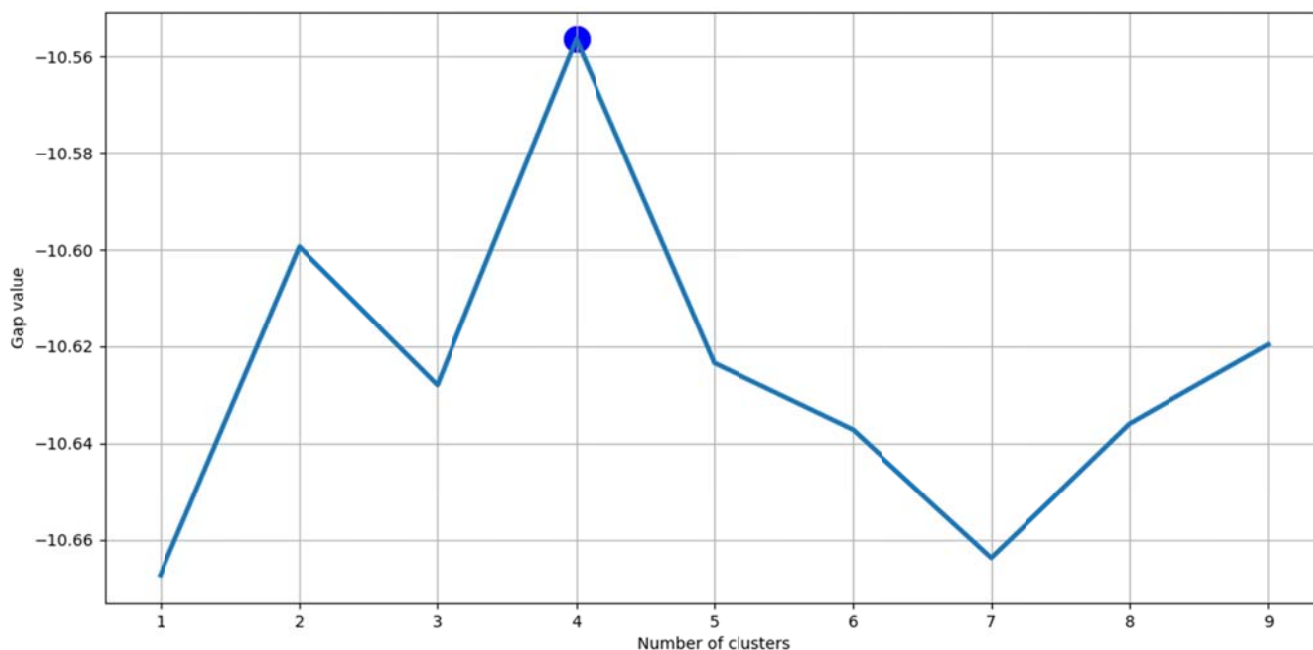


Figure 3: Optimal number of clusters using Gap statistics method

## 2.4 Determination of Optimal Number of Clusters using Silhouette Method

In this section, the Silhouette score technique is used to compute the optimal number of clusters [18,19]. Specifically, the Silhouette coefficient gives insight about the similarities in terms of characteristics of the sensor node around a cluster. The Silhouette coefficient for sensor node  $i$  can be defined as:

$$S_{c(i)} = \frac{d'_{NN}(i) - d_{NN}(i)}{\max\{d'_{NN}(i), d_{NN}(i)\}} \quad (5)$$

where,  $d_{NN}$  denotes the average distance between the location of sensor node  $i$  and other sensor nodes within the same cluster,  $d'_{NN}$  denotes the minimum average distance between the location of sensor node  $i$  and other clusters in which sensor node  $i$  does not belong to. The value of  $S_{c(i)}$  is also defined as;

$$S_{c(i)} = \begin{cases} 1 - \frac{d_{NN}(i)}{d'_{NN}(i)}, & \text{if } d_{NN}(i) < d'_{NN}(i) \\ \frac{d'_{NN}(i)}{d_{NN}(i)} - 1, & \text{if } d_{NN}(i) > d'_{NN}(i) \\ 0, & \text{if } d_{NN}(i) = d'_{NN}(i) \end{cases} \quad (6)$$

The procedure for the computation of the Silhouette coefficient for the set of clusters given in this work is presented in Algorithm 4. The average Silhouette value for every coefficient is obtained from Equation 7;

$$S_{c(\text{avg})} = \text{mean}\{S_{c(i)}\} \quad (7)$$

Where  $S_{c(i)}$  is the Silhouette score.

### Algorithm 4: Determination of Number of Clusters based on Silhouette method

- 1: **Begin**
- 2: Define distance between sensor node and other sensor nodes within its cluster as  $d1$
- 3: Define the minimum distance between sensor node and other clusters not related to it as  $d2$

- 4: Define the average distance between sensor node and other sensor nodes within its cluster as  $d_{NN}(i)$
- 5: Define the average minimum distance between sensor node and other clusters not related to it as  $d'_{NN}(i)$
- 6: Define a set of sensor nodes as  $N$
- 7: Initialize  $d_{NN}(i), d'_{NN}(i)$
- 8: **For each** sensor node  $n$  in  $N$
- 9:     Compute  $d1$
- 10:    Compute  $d2$
- 11:    Compute the distance matrix  $d_{NN}(i) = (d_{NN}(i) + d1)/n$
- 12:    Compute the distance matrix  $d'_{NN}(i) = (d'_{NN}(i) + d2)/n$
- 13: **end For**
- 14: Compute Silhouette coefficient using Equation 3.7
- 15: **end**

For instance, consider a network where 3 clusters are formed, for instance. Each of the clusters contains certain number of distributed sensor nodes. Let cluster  $A$  contain a set of sensors  $\{S_{a1}, S_{a2}, S_{a3}\}$ , cluster  $B$  contains a set of sensors  $\{S_{b1}, S_{b2}, S_{b3}\}$ , and cluster  $C$  contains  $\{S_{c1}, S_{c2}, S_{c3}\}$ . Then,  $a_{a1}(i) = \frac{d_{a12} + d_{a13}}{2}$ , where,  $d_{a12}$  is the distance from  $S_{a1}$  to  $S_{a2}$ , and  $d_{a13}$  is the distance from  $S_{a1}$  to  $S_{a3}$ . Also,  $b_{a1}(i) = \min(d_{ab}, d_{ac})$ , where  $d_{ab}$  denotes the average distance between point  $S_{a1}$  and all sensor points in cluster  $B$ ,  $d_{ac}$  denotes the average distance between point  $S_{a1}$  and all points in cluster  $C$ . Then the Silhouette coefficient for the whole dataset  $S(i)$  is computed as the average Silhouette coefficient of each single point. Now,  $S(i) \rightarrow 0$  implies that the sensor node location is between two clusters;  $S(i) \rightarrow -1$  implies that the sensor node is in the wrong cluster;  $S(i) \rightarrow 1$  implies that the sensor node is in the correct cluster.

A typical result obtain by applying Silhouette method of determination of optimal number of clusters gives the value of four clusters as shown in Figure 4.

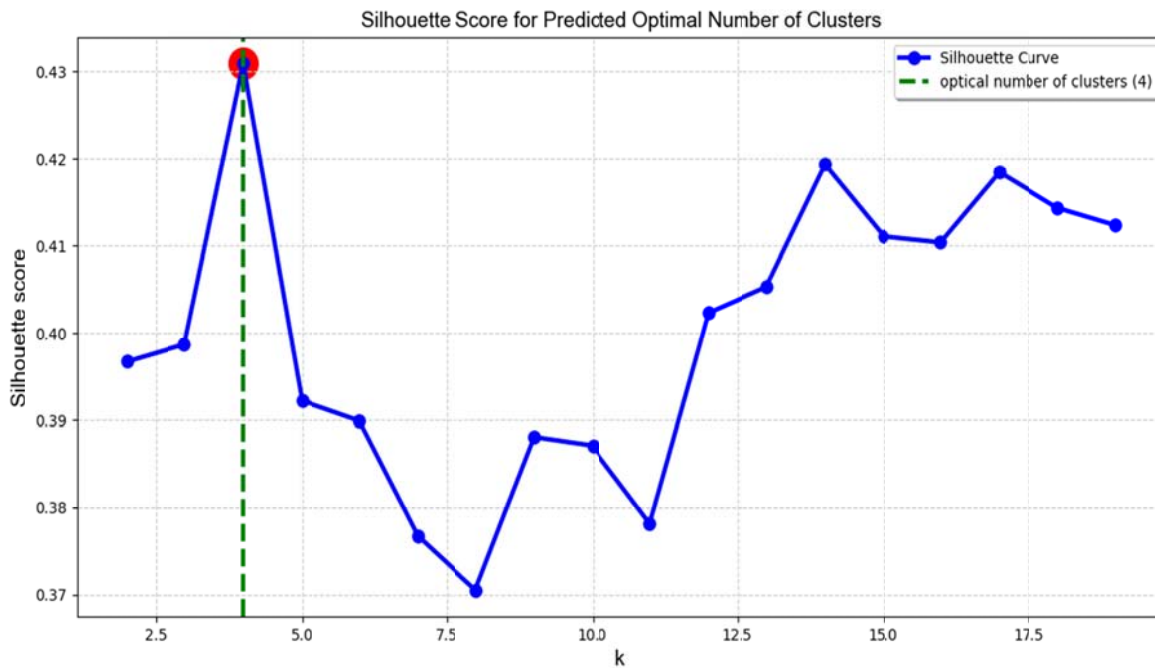


Figure 4: Optimal number of clusters using Silhouette method

Consider the cluster labels for sensor nodes presented in Table 1. *L1* and *L2* belong to cluster 1, while *L3* and *L4* belong to cluster 2. The dissimilarity matrix is presented in

Table 2. The Silhouette coefficient for each of the sensor location must be computed. Similarly, the coefficient must be computed for each of the two clusters as well as the overall cluster

Table 1: Sensor node cluster label

Sensor Location	Cluster Label
<i>L1</i>	1
<i>L2</i>	1
<i>L3</i>	2
<i>L4</i>	2

Table 2: Dissimilarity matrix

Sensor location	<i>L1</i>	<i>L2</i>	<i>L3</i>	<i>L4</i>
<i>L1</i>	0	0.10	0.65	0.55
<i>L2</i>	0.10	0	0.70	0.60
<i>L3</i>	0.65	0.70	0	0.30
<i>L4</i>	0.55	0.60	0.30	0

For *L1*:  $d_{NN}(L1)$  and  $d'_{NN}(L1)$  are computed as:

$$d_{NN}(L1) = \frac{0.1}{1} = 0.1; \quad d'_{NN}(L1) = \frac{0.65 + 0.55}{2} = 0.6$$

By applying Equation 6, the Silhouette coefficient for sensor location  $L1$  can be obtained as:

$$S_{c(L1)} = 1 - \frac{d_{NN}(L1)}{d'_{NN}(L1)} = 1 - \frac{0.1}{0.6} = 0.833$$

For  $L2$ :  $d_{NN}(L2)$  and  $d'_{NN}(L2)$  are computed as:

$$d_{NN}(L2) = \frac{0.1}{1} = 0.1; \quad d'_{NN}(L2) = \frac{0.70 + 0.60}{2} = 0.65$$

By applying Equation 6, the Silhouette coefficient for sensor location  $L2$  can be obtained as:

$$S_{c(L2)} = 1 - \frac{d_{NN}(L2)}{d'_{NN}(L2)} = 1 - \frac{0.1}{0.65} = 0.8461$$

For  $L3$ :  $d_{NN}(L3)$  and  $d'_{NN}(L3)$  are computed as:

$$d_{NN}(L3) = \frac{0.3}{1} = 0.3; \quad d'_{NN}(L3) = \frac{0.65 + 0.70}{2} = 0.675$$

By applying Equation 6, the Silhouette coefficient for sensor location  $L3$  can be obtained as:

$$S_{c(L3)} = 1 - \frac{d_{NN}(L3)}{d'_{NN}(L3)} = 1 - \frac{0.3}{0.675} = 0.5556$$

For  $L4$ :  $d_{NN}(L4)$  and  $d'_{NN}(L4)$  are computed as:

$$d_{NN}(L4) = \frac{0.3}{1} = 0.3; \quad d'_{NN}(L4) = \frac{0.55 + 0.60}{2} = 0.575$$

By applying Equation 6, the Silhouette coefficient for sensor location  $L4$  can be obtained as:

$$S_{c(L4)} = 1 - \frac{d_{NN}(L4)}{d'_{NN}(L4)} = 1 - \frac{0.3}{0.575} = 0.4783$$

The Silhouette Coefficient for cluster 1 is given as:

$$S_{c(1)} = \frac{S_{c(L1)} + S_{c(L2)}}{2} = \frac{0.833 + 0.8461}{2} = 0.8395$$

The Silhouette Coefficient for cluster 2 is given as:

$$S_{c(2)} = \frac{S_{c(L3)} + S_{c(L4)}}{2} = \frac{0.5556 + 0.4783}{2} = 0.5170$$

Finally, the Silhouette coefficient for the entire cluster is

$$\text{given as: } S_c = \frac{S_{c(1)} + S_{c(2)}}{2} = 0.6783$$

## 2.5 Fast Silhouette method

Although silhouette method is a good option for determination of number of clusters, there is room for improvement in terms of computational speed. The major limitation of Silhouette method, just like Elbow method and Gap statistics method is the requirement of initial centroids and number of clusters. The choice of centroid plays significant role in the computational time. In other words, if the clusters are too close or too far apart, then it will require significant amount of time to obtain the result. This is realized by modifying the approach based on the following steps:

- i. **Use Smart Initialization Approach:** Instead of selecting the initial number of cluster randomly, one can apply the probability

distribution approach such that points located far apart can be considered at the initial stage.

- ii. **Scale the Data:** If the characteristic difference between the members of a cluster is significantly large or small, there will be cluster deformation with overrated distance computations. This can be handled by scaling the data within the range of 0 and 1 before loading them to the Silhouette algorithm.
- iii. **Optimize Data Assignment Method:** Silhouette algorithm assigns sensor nodes to the closest centroid based on iterative approach. However, a more effective way of performing this assignment is the triangle inequality where  $\|x\| + \|y\| \geq \|x + y\|$ . This method takes the shortest distance between two points as a straight line, hence, it reduces the distance computation trips while being guided by the distance boundaries

## 2.6 Simulation of the models

In all, the significant problem with Gap statistics method and Silhouette method is that they take a lot of computational resources as well as execution time. Hence, there is a need to develop an Enhanced or fast Silhouette method to address the drawback. Silhouette is selected for optimization since its original implementation has abstraction layer. The Fast Silhouette method was developed and used to determine the optimal number of cluster in the network based on the same design specifications. In order to test and compare the efficiency of the selected methods based on the execution time, each of the methods were experimented while the execution time was monitored. Particularly, Python 3 was used for the simulations of the clustered sensor network with  $N_{nodes} = 5000$ , and network coverage area with  $L(x) = 800m$ ,  $L(y) = 800m$  hence,  $A_{xy} = 800m \times 800m = 640,000 m^2$ .

## 3. RESULTS AND DISCUSSION

### 3.1 Evaluation of Number of Cluster Determination Based on Elbow Method

For the given network space and number of sensor nodes, the result obtained from Elbow method shows that the optimal number of clusters required in the network is four; as shown in Figure 5. Each of the sensors were grouped into *Cluster 0*, *Cluster 1*, *Cluster 2*, and *Cluster 3*

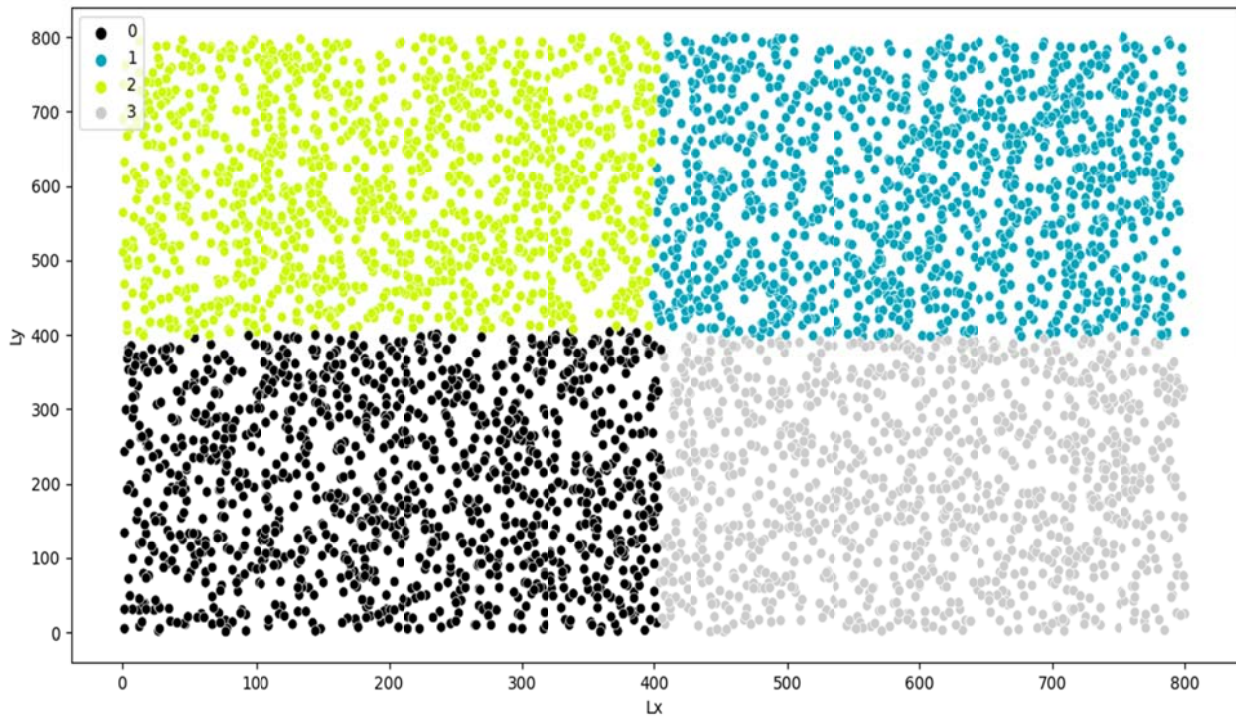


Figure 5: Determination of Number of Cluster Based on Elbow Method

**3.2 Evaluation of Number of Cluster Determination Based on Gap Statistics Method**

Gap statistics was applied to determine the optimal number of cluster in the network based on the design specification.

The result presented in Figure 6 shows that the optimal number of cluster is five. Each of the sensors were grouped into *Cluster 0*, *Cluster 1*, *Cluster 2*, *Cluster 3*, and *Cluster 4*

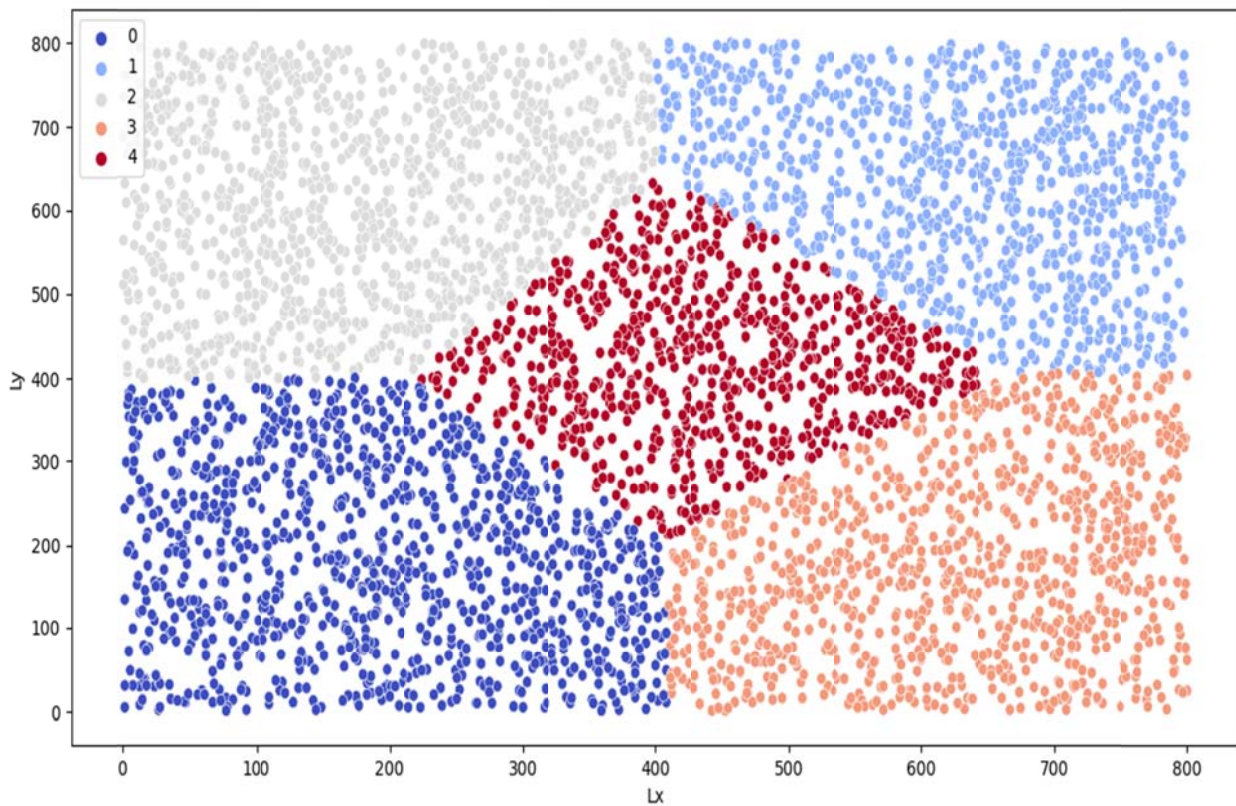


Figure 6: Determination of Number of Cluster Based on Gap Statistics Method

**3.3 Evaluation of Number of Cluster Determination Based on Silhouette Method**

Silhouette method was used to determine the optimal number of cluster in the network based on the design specification. Just as in the case of Gap statistics, the result

presented in Figure 7 shows that the optimal number of cluster is five. Each of the sensors were grouped into

*Cluster 0* , *Cluster 1* , *Cluster 2* , *Cluster 3* , and *Cluster 4*

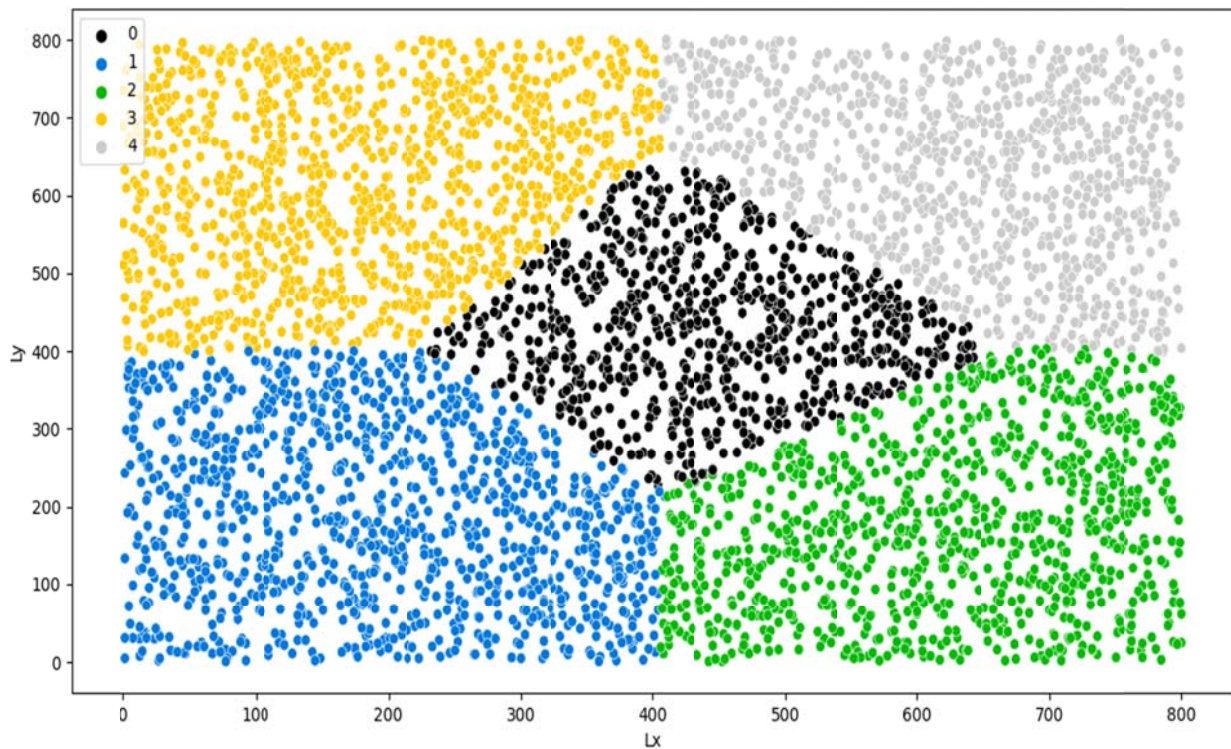


Figure 7: Determination of Number of Cluster Based on Silhouette Method

### 3.4 Evaluation of Number of Cluster Determination Based on the Fast Silhouette Method

The result of the simulation for the lw that the optimal number of cluster is five as shown in Figure 8 and the

computational time is also lower than that of the other methods considered in the study. Each of the sensors were grouped into *Cluster 0*, *Cluster 1*, *Cluster 2*, *Cluster 3*, and *Cluster 4*

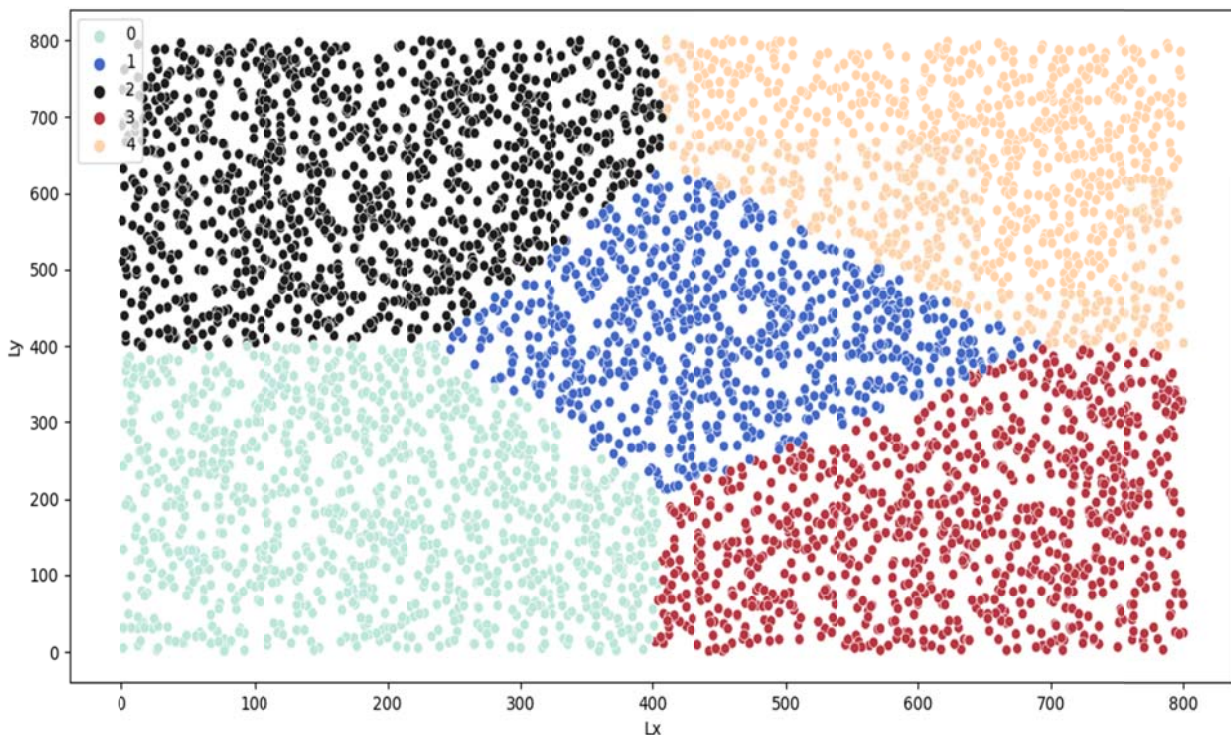


Figure 8: Determination of Number of Cluster Based on Enhanced Silhouette Method

### 3.5 Comparison of Methods used for Determination of Number of Clusters

The methods selected for the determination of optimal number of clusters include: Elbow method, Gap statistics



method, Silhouette method and Fast Silhouette method. From the results presented by Elbow method in Figure 5, the optimal number of clusters was four. However, other methods suggested five as the optimal number of clusters. In order to test and compare the efficiency of the selected methods based on the execution time, each of the methods were experimented while the execution time was monitored. The results for the comparison of the different methods are shown in Figure 8, Figure 9 and Table 3. The results show that the Fast Silhouette method has the lowest

execution time of 2.75 seconds. The results of the normalized model execution time with respect to the Fast Silhouette time (%) show that the Elbow method takes 2.12 % of the time required by the Fast Silhouette, the Gap statistics method takes 1.95% of the time required by the Fast Silhouette, while the classical Silhouette method takes 2.00% of the time required by the Fast Silhouette. In all, the Fast Silhouette method realized the optimal number of cluster with execution time that is about half of the time required by the other three methods studied in this work.

Table 3: The execution time comparison for the various methods used in determining optimal number of clusters

Method	Elbow	Gap statistics	Silhouette	Fast Silhouette
Execution time (s)	5.83	5.37	5.42	2.75
Normalized execution time with respect to the Fast Silhouette execution time (%)	2.12	1.95	2.00	1.00

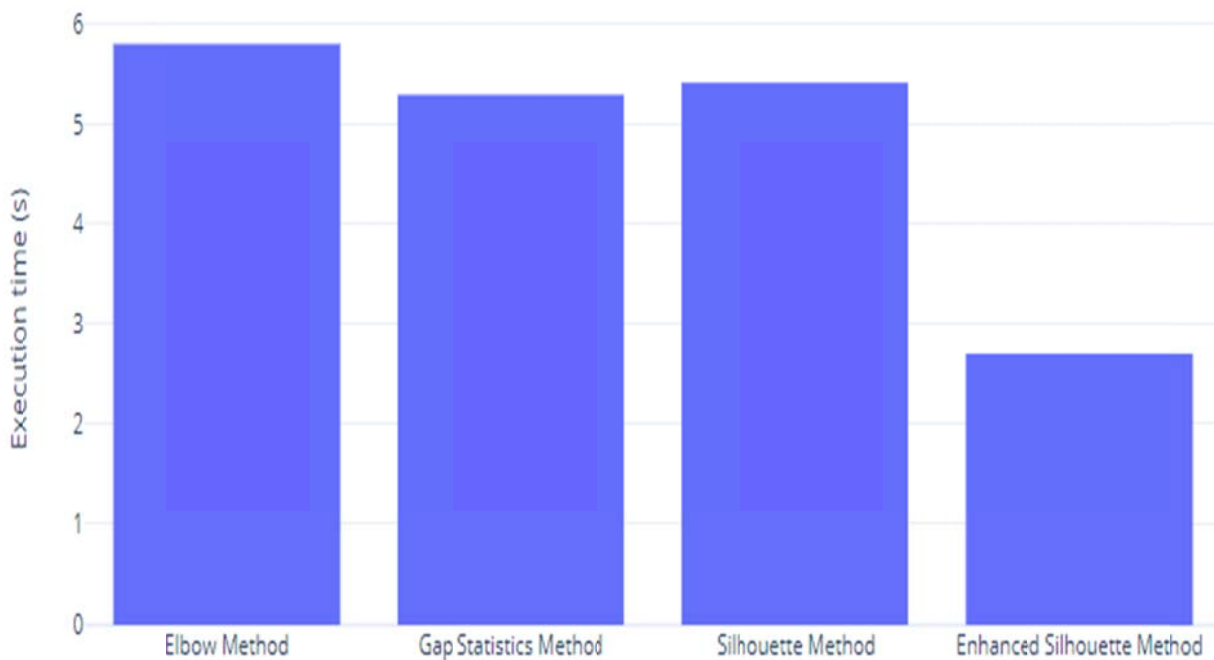


Figure 8: The bar chart of the execution time for the various methods used in determining optimal number of clusters.

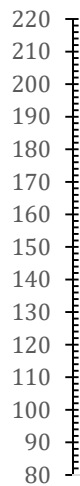


Figure 9: The bar chart of the normalized model execution time with respect to the Fast Silhouette execution time (%)

#### 4. CONCLUSION

In this article, various methods are presented for the determination of the optimal number of clusters required for a given size of sensor network. The methods are Elbow method, Gap statistics method, Silhouette method, and Fast Silhouette method. Although Silhouette method is a good option for determination of number of clusters, there is room for improvement in terms of computational speed. Notably, the major limitation of Silhouette method, just like Elbow method and Gap statistics method is the requirement of initial centroids and number of clusters. The choice of centroid plays significant role in the model computational time. In other words, if the clusters are too close or too far apart, then it will require significant amount of time to obtain the result. The drawback is addressed in this work by the development of the Fast silhouette method which reduced the computation time by approximately 50 %. The improvement is demonstrated through a simulated experiment conducted using Python 3.

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